Reasoning with geometric shapes

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Geometry belongs to branches of mathematics that develop students' visualisation, intuition, critical thinking, problem solving, deductive reasoning, logical argument and proof (Jones, 2002). It provides the basis for the development of spatial sense and plays an important role in acquiring advanced knowledge in science, technology, engineering, and mathematics. The *Australian Curriculum: Mathematics* (Australian Curriculum Assessment and Reporting Authority (ACARA), n.d) emphasises the need to help children develop an increasingly sophisticated understanding of geometric ideas, to be able to define, compare and construct figures and objects, and to develop geometric arguments. This article will look at some of the issues involved in the teaching and learning of two-dimensional shapes and illustrate how activities such as paper-folding tasks can be used to encourage visualisation and geometric reasoning.

The nature of geometry

Geometry deals with abstract concepts in that the natural world we live in is not made up of exact examples of geometrical shapes. The outline of a pine tree may look like a triangle, but a nearer view reveals it not to be. Even the full moon loses its circularity when observed closely. Instead, instructional representations such as shapes and solids in the world of geometry symbolise the 'ideal', 'perfect' shapes with exact relationships that can be studied in terms of their invariance, symmetry and transformation (Johnston-Wilder & Mason, 2005; Jones, 2002). Invariance deals with the properties of a configuration that remain unchanged under a set of transformations. For example, regardless of the types of quadrilaterals, their interior angles always add to 360˚. Symmetry looks at the process by which a shape is transformed into a new one yet retains certain aspects of its property. Transformation permits students to comprehend concepts of congruence and similarity, ratio and proportion. To nurture an understanding of symmetry and transformation processes, which in turn helps develop the idea of invariance, requires an ability to visualise what will occur.

Visualisation as a tool for geometric reasoning

Visualisation is the ability to mentally manipulate, rotate, twist, or invert representations such as a figure or object (McGee, 1979). This ability is dependent on two factors. The first relates to the purpose and design of the representations. Some representations act as general illustrations of a shape,

for example, a shape with three straight sides. Others refer specifically to a particular shape, for example, a triangle with the measurement of 40°, 60°, and 80°. If a representation has too much information, the geometrical relationships will be obvious to the children and prevent them from developing geometrical reasoning ability. Conversely, if a representation has too little information, children may not be able to understand the relationships and complete a geometric task. Thus, an important goal in education is to design a representation that can stimulate children to appreciate the multiple relationships involved in geometric activities. A second consideration concerns the way in which a representation is positioned in relation to the viewer. To accurately visualise the relationships a representation seeks to portray, the viewer must be able to 'see' the figure in his/her mind and interpret the information based on his/her knowledge of geometric properties, which in turn is consolidated through knowing the language and definitions of 2D shapes.

The language of geometry

Language plays a critical role in the development of geometric thought. Many words and names used in geometry are taken from Greek and Latin. Inconsistency in the way both languages are used to name shapes can lead to misconceptions (Booker, Bond, Sparrow, & Swan, 2014). For example, polygon names such as pentagon, heptagon, octagon and nonagon, use Greek numbers as prefixes to mean *n*-angled shapes. Conversely, the words triangle and quadrilateral are from Latin and mean 'three-angled' and 'four-sided' respectively (Figure 1).

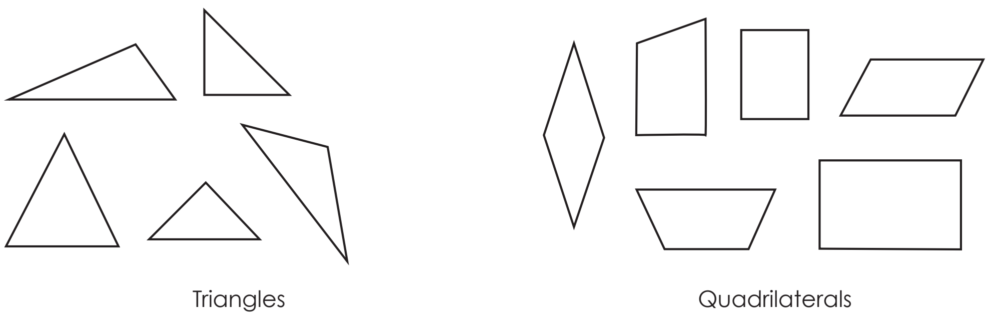


Figure 1. Examples of triangles and quadrilaterals.

The way definitions of polygons are written can also create difficulties for many learners. A definition found in a dictionary describes the meaning of a word,phrase or symbol. A mathematical definition is more than a description of meaning. It must include only terms previously defined or specifically designated as undefined and cannot have contradictory meaning (Usiskin & Griffin, 2008). For example, if a triangle is defined as a 'three-angled shape', the expectation would be that the words 'three', 'angled' and 'shape' have been explicitly defined previously so that their meanings are known.

In mathematics, definitions of shapes are used to identify, distinguish and classify one from another. Many people find learning the definitions and hierarchical classification of quadrilaterals difficult and confusing (Fujita & Jones, 2007). For example, a square is a rectangle because the word 'rectus' in Latin means right or straight and 'angle' means small bend. The use of the word trapezium further confuses the situation. It has two contradictory meanings. Outside of the USA, it is a quadrilateral with one pair of parallel sides, while in the USA this is known as a trapezoid and a trapezium is a quadrilateral with no

parallel sides. The word trapezoid comes from the Greek word 'trapeza' meaning table and the suffix '-oid' meaning resembling. It may come from the realisation that when a table is viewed from the front, it looks like a shape with the sides of its front and back parallel but not its other sides (Usiskin & Griffin, 2008).

This type of confusion may be because many people assume that the language of mathematics is universal, set in stone. In reality, definitions are written by people using different words, some are inclusive, others exclusive. An inclusive definition for a trapezium is a shape with one pair of parallel sides. An exclusive definition is a shape with only one pair of parallel sides. The application of this difference is only realised when learning the trapezium rule in calculus. Given that learning mathematics is about developing ways of thinking, learning these names and definitions by rote is pointless. Instead, children should be given opportunities to explore and engage in activities that can bring out the need to name and define these shapes, and comprehend how mathematicians make decisions about how to organise mathematical knowledge.

Children's understanding of 2D shapes

Children's early experience with geometric shapes often involves playing and sorting out pattern blocks. They learn to name each shape without necessarily knowing its properties. Examining a pile of pattern blocks easily reveals that many shapes are absent. Much of the constructed environment they live in abounds in rectangular and triangular shapes. Other types of polygons are less prominent. Children quickly develop a stereotypical idea of how shapes should look. To a child, a rectangle is a shape with two long and two short sides. A square has four straight sides with the base sitting horizontally on the plane (□). When the square is tilted (◇), many children think it is a rhombus or diamond. Children also assume that polygons such as pentagons, hexagons and octagons have equal sides (Figure 2). They have difficulty accepting that an irregular five-angled shape is a pentagon and will have problems naming seven-, nine- and ten-angled shapes. In essence, when children talk about geometric shapes, they are describing what they saw, the attributes rather than the properties of these shapes.

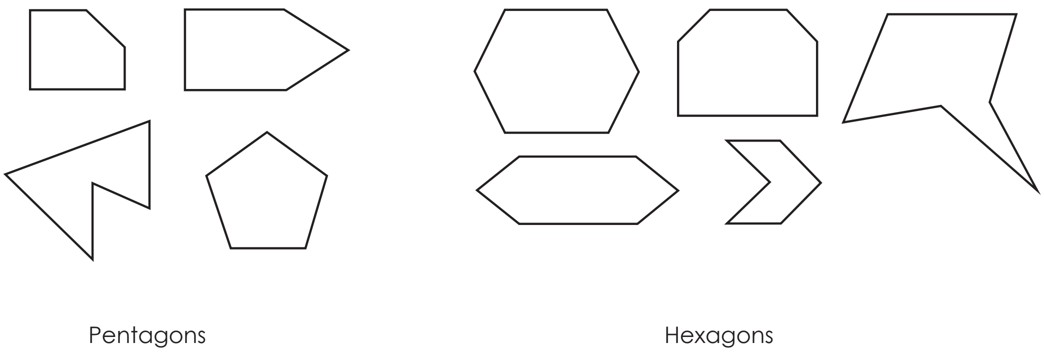


Figure 2. Examples of pentagons and hexagons.

In contrast, paper-folding activities, where children construct their own shape, can be a great tool to help them comprehend geometric ideas and alleviate the linguistic confusion. It is low cost and has been widely used to develop geometric reasoning and problem solving skills in countries such as

Japan, the United Kingdom and Turkey. It provides the flexibility that concrete objects do not and sensitises students to the names and properties of geometric shapes.

Folding triangles

Give children some square papers. Investigate how many different ways to fold a triangle (Figure 3). How many types of triangles can they fold? Have children fold the shape and write down the number of folds. The purpose here is to introduce the idea of making corners. Get children to compare and see if all the corners are the same. Ask children to unfold the paper and trace the crease lines made by the folds. What shape did the crease lines make? Have children shade the regions using different colours to highlight the shapes.

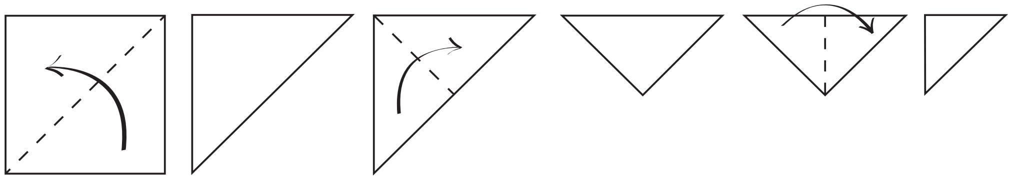


Figure 3. Instructions for folding isosceles right-angled triangles using a square paper.

Now use metric papers such as A4 or A5. Would there be any difference in the way triangles are folded? Some children will notice that one fold on a square paper can make a right-angled isosceles triangle whereas three folds are needed on a metric paper (Figure 4). Folding other types of triangles sensitises children to the idea of creating three 'pointy bits'. Get children to compare and decide which shape has the largest corner. How would they organise their triangles and how should they report their findings? If working with younger children, talk about a triangle as a shape with three corners. Formal terms may be used for older children.

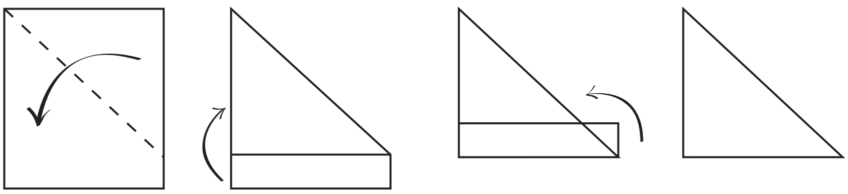


Figure 4. Instructions for folding isosceles right-angled triangles using a metric paper.

Folding squares

Similar instructional sequences can be given to teach the concept of squares and rectangles. Folding both shapes using square papers is fairly straightforward. Folds can be made horizontally or vertically. Using direct comparison reinforces the idea that both shapes have the same kind of corners (Figure 5). Folding squares using metric papers taps into earlier learned ways of folding triangles, reinforcing the idea that a square can be made with two identical (congruent) right-angled triangles.

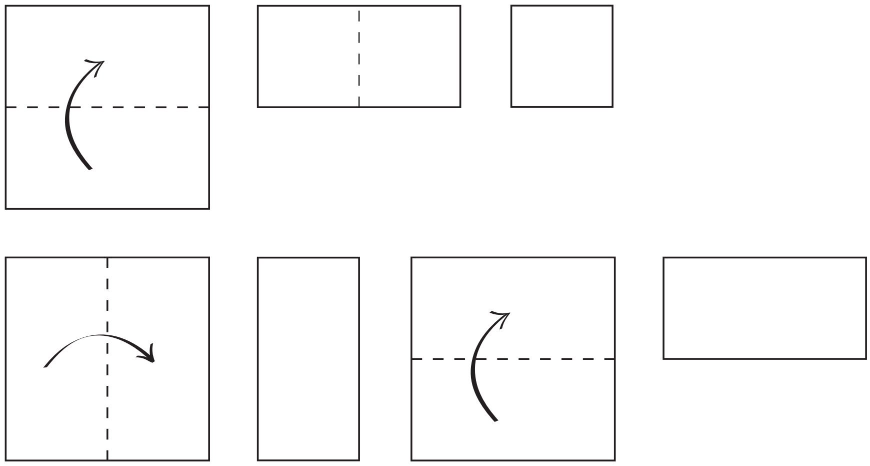


Figure 5. Examples of using square papers to fold squares and rectangles.

Folding polygons

Investigate how many different types of shapes can be created using just one fold. Using square papers, have children create different shapes by making only one fold. Compare and talk about how they would name the shapes they created. Some children may name the shapes according to the number of angles (corners), for example, 4-angled, 5-angled and so on. There are four different ways to fold a triangle (Figure 6). One fold can also make four-, five- and nine-sided shapes but not seven- or eight-sided shapes. Introduce the formal terms and lead children to comprehend that polygons mean many angled shapes. Now do the same task using metric papers. What do they notice?

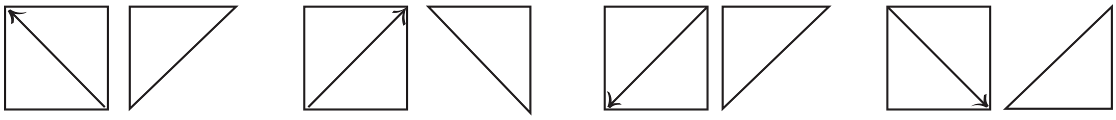
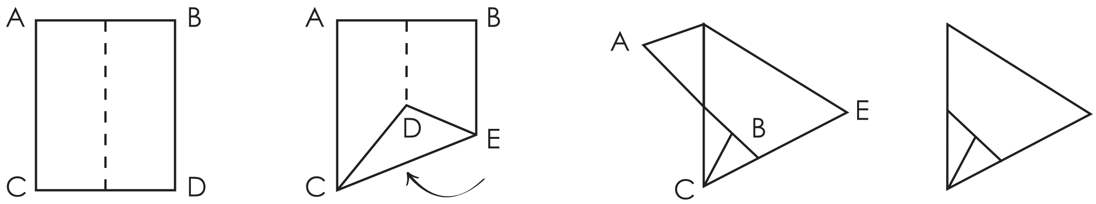
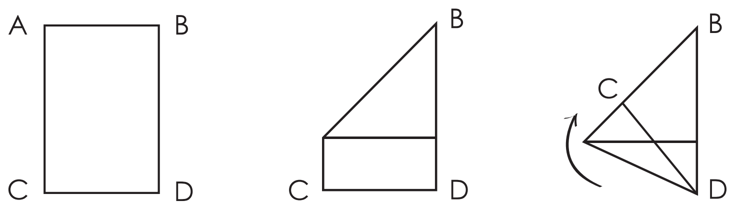


Figure 6. Four different ways of folding a triangle.

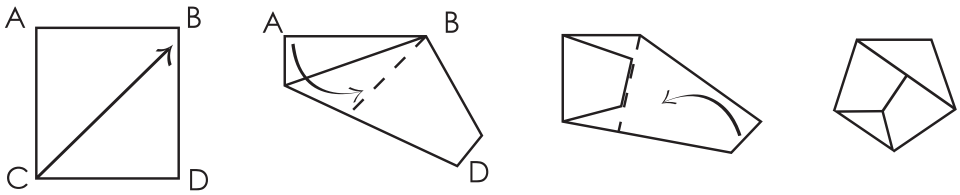
Next, investigate how to fold other polygons by making multiple folds. Is it possible to fold an arbitrary polygon so that it has all of its corners the same and all of its sides the same length? How would they name these types of polygons? Instructions for folding equilateral triangles, pentagons and hexagons can be found in Figure 7 using metric papers. Due to the properties of regular polygons, not all of them can be done through folding. Nevertheless, folding certain regular polygons helps children generalise the idea that polygons are named by the number of angles the shape has.



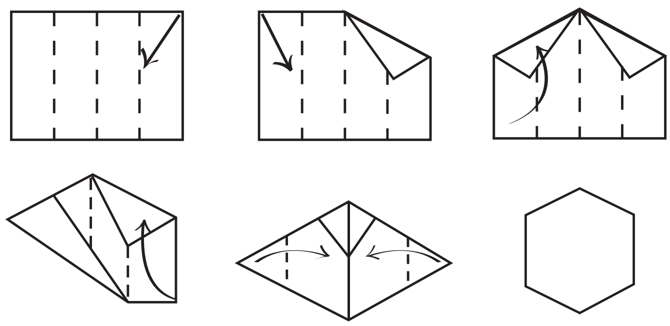
Equilateral triangles



Isosceles triangle that has no right angle



Regular pentagon

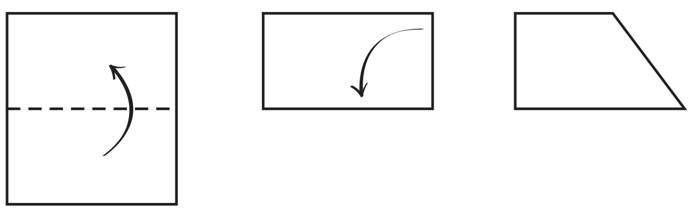


Regular hexagon

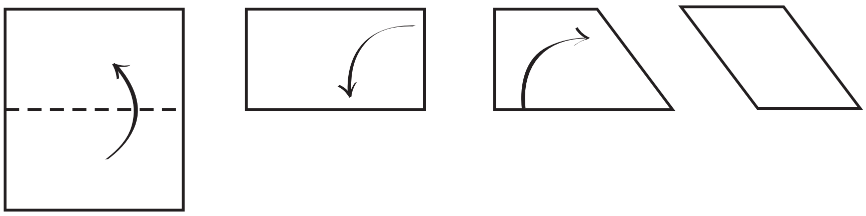
Figure 7. Instructions for folding regular polygons.

Types of quadrilaterals

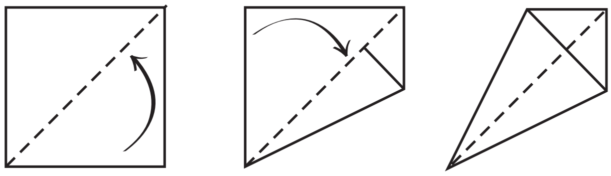
Older children can investigate how many different types of four-sided shapes they can create (Figure 8). Have children describe, compare and name the difference or sameness among these shapes. Compare their descriptions with the formal definitions. Can they make other quadrilaterals that share the same criteria? Discuss how they will put them into groups (Figure 9). For example, can children group the shapes according to angles and the number of parallel lines?



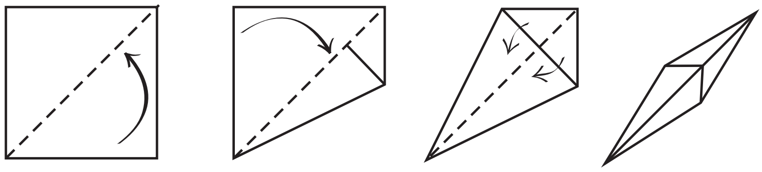
Trapezium



Parallelogram



Kite



Rhombus

Figure 8. Instructions for folding trapezium, parallelogram and kite using square papers.

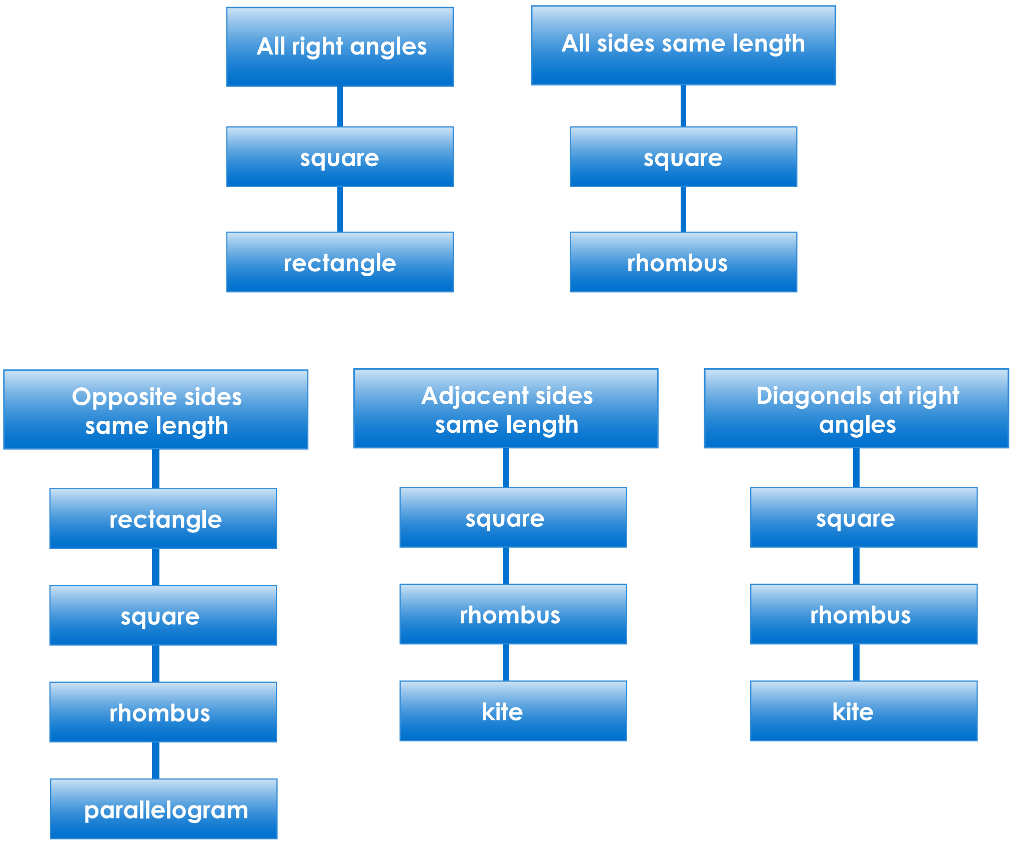


Figure 9. Some properties of quadrilaterals.

Conclusion

Geometry is a wonderful subject to learn and teach. It appeals to our visual, aesthetic and intuitive senses and is intimately connected with learning advanced mathematics, applicable to engineering, industrial design, and medical science (TED, 2008). Visualisation is crucial in helping children understand shapes and their properties. This ability is dependent on both the design of the representation as well as an individual's existing network of beliefs, experiences and understanding. Paper folding activities sensitise children to the properties of shapes. Further, viewing shapes from different orientations removes children's stereotypic understanding of how certain shapes look,thus supporting the understanding of essential geometric principles.

References

Australian Curriculum Assessment and Reporting Authority (ACARA). (n.d). *The Australian Curriculum: Mathematics.* http://www.australiancurriculum.edu.au/

Booker, G., Bond, D., Sparrow, L., & Swan, P. (2014). *Teaching primary mathematics (5th ed.).* Frenchs Forest, NSW: Pearson Prentice Hall.

Fujita, T., & Jones, K. (2007). Learners' understanding of the definitions and hierarchical classification of quadrilaterals: Towards a theoretical framing. *Research in Mathematics Education, 9*(1), 3-20.

Johnston-Wilder, S., & Mason, J. (2005). *Developing thinking in geometry.* London: The Open University.

Jones, K. (2002). Issues in the teaching and learning of geometry. In L. Haggarty (Ed.), *Aspects of teaching secondary mathematics: Perspectives on practice* (pp. 121-139). London: RoutledgeFalmer.

McGee, M. (1979). Human spatial abilities: Psychometric studies and environmental, genetic, hormonial and neurological influences. *Psychological Bulletin, 86*, 889-918.

TED (Producer). (2008, 15th March 2014). Robert Lang: *The math and magic of origami.* Retrieved from http://www.ted.com/talks/robert_lang_folds_way_new_origami?language=en

Usiskin, Z., & Griffin, J. (2008). *The classification of quadrilaterals: A study of definition.* Charlotte, NC: Information Age Publishing, Inc.